Eur. Phys. J. C 42, 303–308 (2005)
Digital Object Identifier (DOI) 10.1140/epjc/s2005-02288-7

THE EUROPEAN
PHYSICAL JOURNAL C

A universal sum rule for hadronic elastic scattering with applications to the Tevatron, RHIC and LHC

G. Pancheri[1,a], Y. Srivastava[2,3,b], N. Staffolani[2,c]

[1] Laboratori Nazionali di INFN, Frascati, Italy
[2] Dipartimento di Fisica and INFN, Università di Perugia, Italy
[3] Physics Department, Northeastern University, Boston MA, USA [**Author please insert the full addresses**]

Received: 15 August 2004 / Revised version: 27 April 2005 /
Published online: 8 July 2005 –

Abstract. Under the assumption that at high energies total absorption prevails so that the imaginary part of the scattering amplitude dominates, we present a sum rule for all hadronic elastic differential cross-sections. We find that the dimensionless quantity $\frac{1}{2}\int(\mathrm{d}t)\sqrt{\frac{1}{\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}t}} \to 1$, at asymptotic energies. A comparison with experimental data from ISR and Tevatron confirms a trend towards its saturation and some estimates are presented for LHC. Its universality and further consequences for the nature of absorption in QCD based models for elastic and total cross-sections are explored.

1 Introduction

Ab initio calculations of hadronic elastic amplitudes and total cross-sections in QCD are presently difficult due to our meager understanding of "soft" physics, that is, the non-perturbative and confinement region of QCD. Hence, the need to invoke general principles such as analyticity and unitarity to obtain bounds and restrictions on these amplitudes as pioneered by Froissart and Martin [1–3]. Analyticity and unitarity are expected to hold for finite-ranged hadron dynamics, only massive hadrons being the bound states of quarks and glue. The central result of our work described below is that under rather mild assumptions, a universal behavior for all hadrons is likely to emerge at asymptotic energies. It is exhibited as a sum rule which should become exact at infinite energy. The physics behind the sum rule is that hadrons are not elementary particles and hence there exists a tight correlation between the rate of growth with energy of the elastic amplitude and its fall off with the momentum transfer.

This paper is organized as follows. In Sect. 2, we briefly review the eikonal formalism. The eikonal formalism serves a dual purpose: it not only incorporates the fact that at high energies hadronic elastic amplitudes are concentrated at small scattering angles (i.e. the elastic amplitudes show a fast decrease with momentum transfer) but it in addition guarantees that the direct (s-)channel unitarity bound is not superseded. In Sect. 3, a discussion of generic models is made in order to provide a background and to motivate the central result. Section 4 deals with some rigorous constraints which the elastic amplitudes must obey in the eikonal picture. In Sect. 5 we present the sum rule and its experimental substantiation. We also discuss here how the assumptions made to obtain the sum rule may be further relaxed. In the concluding section, Sect. 6, we discuss the universality and extensions of the sum rule and further consequences.

2 Eikonal formalism for the elastic amplitude

Consider the amplitude for an elastic process $A(p_a) + B(p_b) \to A(p_c) + B(p_d)$. Let $s = (p_a + p_b)^2$ be the square of the CM energy; let $t = (p_a - p_c)^2 = -2k^2(1 - \cos\theta)$ be the momentum transfer and k its CM 3-momentum in the s-channel. Let us normalize the amplitude so that the differential and total cross-sections are given by[1]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \pi|F(s,t)|^2; \quad \sigma_{\mathrm{tot}}(s) = 4\pi\mathrm{Im}F(s,t=0). \quad (2.1)$$

We may expand the scattering amplitude in terms of the partial-wave phase shifts as follows:

$$F(k,\theta) = \left(\frac{\mathrm{i}}{2k^2}\right)\sum_l (2l+1)[1-\eta(l,s)\mathrm{e}^{2\mathrm{i}\delta_{\mathrm{R}}(l,s)}]P_l(\cos\theta), \quad (2.2)$$

[a] e-mail: Giulia.Pancheri@lnf.infn.it
[b] e-mail: Yogendra.Srivastava@pg.infn.it
[c] e-mail: Nicola.Staffolani@pg.infn.it

[1] Since our applications are for large s, we ignore hadronic masses in the kinematics where ever possible. For example, $4k^2$ is approximated to s, the lower limit on t is extended to $-\infty$, etc.

where $\delta_{\mathrm{R}}(l,s)$ is the real part of the phase shift and the inelasticity factor η is related to the imaginary part of the phase shift δ_{I} via

$$\eta(l,s) = \mathrm{e}^{-2\delta_{\mathrm{I}}(l,s)} \quad (0 \le \eta \le 1). \tag{2.3}$$

For small angles, the momentum transfer is $\sqrt{-t} = q = (k\theta)$. For small angles and high k, the impact parameter b is defined as $bk = (l + 1/2)$. In this limit, the Legendre functions $P_l(\cos\theta)$ are approximated by the Bessel functions

$$P_l(\cos\theta) \to J_0(bq), \tag{2.4}$$

through which one arrives at the eikonal expansion

$$F(s,t) = \mathrm{i}\int_0^\infty b\mathrm{d}b J_0(b\sqrt{-t})\tilde{F}(s,b) \tag{2.5}$$

with the "b-wave" amplitudes

$$\tilde{F}(s,b) = 1 - \eta\mathrm{e}^{2\mathrm{i}\delta_{\mathrm{R}}}. \tag{2.6}$$

In the above and the following we shall (wherever not explicitly needed) suppress the dependence of η and δ_{R} on the variables s and b.

Using (2.1)–(2.6), the total, elastic and the inelastic cross- sections are given by

$$\begin{aligned}\sigma_{\mathrm{tot}}(s) &= (4\pi)\int_0^\infty b\mathrm{d}b[1 - \eta\cos(2\delta_{\mathrm{R}})] \\ &= (4\pi)\int_0^\infty b\mathrm{d}b F_{\mathrm{T}}(b,s).\end{aligned} \tag{2.7}$$

$$\begin{aligned}\sigma_{\mathrm{el}}(s) &= (2\pi)\int_0^\infty b\mathrm{d}b[\{1 - \eta\cos(2\delta_{\mathrm{R}})\}^2 \\ &\qquad + (\eta\sin(2\delta_{\mathrm{R}}))^2] \\ &= (2\pi)\int_0^\infty b\mathrm{d}b F_e(b,s).\end{aligned} \tag{2.8}$$

$$\sigma_{\mathrm{in}}(s) = (2\pi)\int_0^\infty b\mathrm{d}b[1 - \eta^2]. \tag{2.9}$$

Equations (2.5) and (2.6) can be inverted to give the complex "b-wave" amplitudes as

$$\begin{aligned}&[(1 - \eta\cos(2\delta_{\mathrm{R}})) - \mathrm{i}\eta\sin(2\delta_{\mathrm{R}})] \\ &= -\mathrm{i}\int_0^\infty q\mathrm{d}q J_0(bq)F(s,t).\end{aligned} \tag{2.10}$$

Note that if one defines the average number of collisions as

$$\mathrm{e}^{-n/2} = \eta\cos(2\delta_{\mathrm{R}}), \tag{2.11}$$

then

$$\sigma_{\mathrm{tot}}(s) = (4\pi)\int_0^\infty b\mathrm{d}b[1 - \mathrm{e}^{-n/2}], \tag{2.12}$$

and the elastic cross-section reads

$$\begin{aligned}&\sigma_{\mathrm{el}}(s) \\ &= (2\pi)\int_0^\infty b\mathrm{d}b[(1 - \mathrm{e}^{-n/2})^2 + \mathrm{e}^{-n}\tan^2(2\delta_{\mathrm{R}})].\end{aligned} \tag{2.13}$$

The forward differential cross-section reads

$$\left[\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right]_{t=0} = \frac{\sigma_{\mathrm{T}}^2}{16\pi}[1 + \rho^2(s,0)], \tag{2.14a}$$

where the "ρ-parameter" denotes as usual the ratio between the real and the imaginary part of the forward amplitude

$$\rho(s,0) = \mathrm{Re}F(s,0)/\mathrm{Im}F(s,0). \tag{2.14b}$$

For the slope parameter $B(s)$, we need to expand the Bessel function for small t. We find

$$\begin{aligned}B(s) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\ln\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right]_{t=0} \\ &= \frac{2\,[\mathrm{Re}F(s,0)\mathrm{Re}F_2(s,0) + \mathrm{Im}F(s,0)\mathrm{Im}F_2(s,0)]}{|F(s,0)|^2}.\end{aligned} \tag{2.15}$$

The second moment factors $\mathrm{Re}F_2(s,0)$ and $\mathrm{Im}F_2(s,0)$ are given through the real and the imaginary parts of

$$\begin{aligned}&F_2(s,0) \\ &= \frac{\mathrm{i}}{4}\int b\mathrm{d}b b^2[(1 - \eta\cos(2\delta_{\mathrm{R}})) - \mathrm{i}\eta\sin(2\delta_{\mathrm{R}})].\end{aligned} \tag{2.16}$$

Thus, the total slope $B(s)$ is the sum of the slopes from the real and the absorptive parts. The main contribution is from the absorptive part which is given by

$$\begin{aligned}&B_{\mathrm{absorptive}}(s) \\ &= \left(\frac{1}{2[1 + \rho^2(s,0)]}\right)\frac{\int b\mathrm{d}b b^2[1 - \eta\cos 2\delta_{\mathrm{R}}]}{\int b\mathrm{d}b[1 - \eta\cos 2\delta_{\mathrm{R}}]},\end{aligned} \tag{2.17}$$

augmented by a small contribution from the real part of the amplitude

$$\begin{aligned}&B_{\mathrm{real}}(s) \\ &= \left(\frac{\rho(s,0)}{2[1 + \rho^2(s,0)]}\right)\frac{\int b\mathrm{d}b b^2\eta\sin 2\delta_{\mathrm{R}}}{\int b\mathrm{d}b[1 - \eta\cos 2\delta_{\mathrm{R}}]}.\end{aligned} \tag{2.18}$$

Since $\rho(s,0)$ is indeed rather small at high energies, the effective slope $B_0(s)$ is essentially due to the absorptive amplitude alone (that is if the real part of the phase shifts were neglected)

$$B_0(s) = \frac{2\mathrm{Im}F_2(s,0)}{\mathrm{Im}F(s,0)} = (1/2)\langle b^2\rangle. \tag{2.19}$$

A first rough estimate [4] of $B_0(s)$ is obtained under the assumption that

$$\left[\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right] \approx \left[\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right]_{t=0}\mathrm{e}^{B_0(s)t}. \tag{2.20}$$

Integrating (2.20) and employing (2.14b), one finds that

$$B_0(s) \approx \left(\frac{\sigma_{\mathrm{tot}}(s)}{16\pi}\right)\left(\frac{\sigma_{\mathrm{tot}}(s)}{\sigma_{\mathrm{el}}(s)}\right)[1 + \rho^2(s,0)], \tag{2.21}$$

which is fairly close to the experimental values at high energies. Below we investigate $B_0(s)$ in generic models.

3 Generic model dependent results

Let us for simplicity set the real part of the phase shift $\delta_{\mathrm{R}} \approx 0$, so that $F_e = F_{\mathrm{T}}^2$ as defined in (2.7) and (2.8). Then dimensional analysis tells us that if at large s, there is only one scale, i.e., if $F_{\mathrm{T}}(b,s) = f(b/b_{\max}(s))$,

$$\sigma_{\mathrm{tot}}(s) = [4\pi b_{\max}^2(s)] \int_0^\infty x\mathrm{d}x f(x);$$
$$\sigma_{\mathrm{el}}(s) = [2\pi b_{\max}^2(s)] \int_0^\infty x\mathrm{d}x f^2(x), \tag{3.1}$$

so that their ratio

$$R_{\mathrm{el}}(s) = \sigma_{\mathrm{el}}(s)/\sigma_{\mathrm{tot}}(s) = \frac{\int_0^\infty x\mathrm{d}x f^2(x)}{2\int_0^\infty x\mathrm{d}x f(x)} \tag{3.2}$$

is independent of s. Also the slope parameter $B_0(s)$ would scale as $b_{\max}^2(s)$.

While the above would not be satisfactory from the point of view of obtaining the observed experimental variations in s for the ratio in (3.2), it would be instructive to evaluate it analytically in some simple models as limiting cases of more realistic models.

(i) Gaussian distribution

Let

$$f(x) = f_0^{\mathrm{G}} \mathrm{e}^{-x^2} \quad (f_0^{\mathrm{G}} \leq 1), \tag{3.3}$$

then

$$R_{\mathrm{el}}^{\mathrm{Gauss}}(s) = (1/4) f_0^{\mathrm{G}} \leq 1/4. \tag{3.4}$$

(ii) Exponential or Boltzmann distribution

Let

$$f(x) = f_0^{\mathrm{B}} \mathrm{e}^{-x} \quad (f_0^{\mathrm{B}} \leq 1), \tag{3.5}$$

then

$$R_{\mathrm{el}}^{\mathrm{Boltzmann}}(s) = (1/8) f_0^{\mathrm{B}} \leq 1/8. \tag{3.6}$$

(iii) Fermi distribution

Let

$$f(x) = f_0^{\mathrm{F}} \frac{2}{\mathrm{e}^x + 1} \quad (f_0^{\mathrm{F}} \leq 1), \tag{3.7}$$

then

$$R_{\mathrm{el}}^{\mathrm{Fermi}}(s) = f_0^{\mathrm{F}}[1 - (\ln 2)/D], \tag{3.8}$$

where

$$D = \int_0^\infty \frac{x\mathrm{d}x}{\mathrm{e}^x + 1} = \sum_{n=0}^{\infty} [(-)^n/(n+1)^2] = \pi^2/12. \tag{3.9}$$

Thus,

$$R_{\mathrm{el}}^{\mathrm{Fermi}}(s) = f_0^{\mathrm{F}} \left[1 - \frac{12 \ln 2}{\pi^2}\right] \approx 0.15723 f_0^{\mathrm{F}} \leq 0.16. \tag{3.10}$$

(iv) Limiting "Froissart" distribution

To saturate the Froissart limit, one essentially need assume that all partial waves are completely absorbed up to an $l \leq l_{\max}(s)$ and beyond that the partial-wave amplitudes are zero. In impact parameter language, it translates into (see (2.11))

$$n(b,s) \to \infty \text{ for } b \leq b_{\max}(s) \approx b_0(\ln s);$$
$$n(b,s) \to 0 \text{ for } b \geq b_{\max}(s), \tag{3.11}$$

so that (through (2.2), (2.3) and (2.11))

$$[1 - \mathrm{e}^{-n(b,s)/2}] = 1 \text{ for } b \leq b_{\max}(s);$$
$$[1 - \mathrm{e}^{-n(b,s)/2}] = 0 \text{ for } b \geq b_{\max}(s). \tag{3.12}$$

In such a model

$$R_{\mathrm{el}}^{\mathrm{Froissart}}(s) = 1/2. \tag{3.13}$$

The significance of the above four models becomes clear when one compares the experimental data on R_{el} at high energies with predictions from each of the above. Experimentally [5] $R_{\mathrm{el}}(p\bar{p})$ rises from about 0.17 at $\sqrt{s} = 62\,\mathrm{GeV}$ to about 0.25 at $\sqrt{s} = 1.8\,\mathrm{TeV}$. In this high energy region, the Fermi distribution gives too low a value, whereas the limiting Froissart distribution gives too high a value for the elastic to total cross-section ratio.

The moral to be drawn from the above analytic models is that the Fermi and Froissart distributions are not adequate, whereas the Boltzmann distribution is closer to reality for moderately large energies which should crossover to a Gaussian distribution at yet higher energies. This points to the need for generic 2-component models for $n(b,s)$ [6–8,10,9] with a "soft" and a "hard" part. In the total cross-section, the soft component asymptotes to a constant as a function of energy, whereas the hard part incorporates the rise with energy. It would appear that the dominant component at medium energies (the "soft" part) should have Boltzmann (or thermal) fall off for large b, whereas the component dominant at truely large energies (the "jet" part) should have a Gaussian fall off in b. This is confirmed by detailed analyses [11,12].

A few words about f_0, that is, the value of $n(b,s)$ at $b = 0$. In the model of [7,11,12] for example, f_0 is extremely close to 1 for all large s. This value is of direct interest for our sum rule described in Sect. 5.

4 Analyticity requirements on the impact parameter distribution

The requirements of analyticity impose restrictions on the large b-behavior of the function $n(b,s)$. Let us briefly examine them through the expressions given in Sect. 2.

The finite range of hadronic interactions implies that the partial-wave expansion converges beyond the physical region, i.e., throughout the Lehmann ellipse. This requires that $F_{\mathrm{el}}(s,t)$ be analytic in t up to $t = 4\mu^2$, where μ

is the pion mass. For positive t, we continue the above expression for $\sqrt{-t} = \mathrm{i}W$, with W real and positive. In this "unphysical" region, we have

$$F(s, W^2) = \mathrm{i} \int b\mathrm{d}b I_0(bW)\tilde{F}(s,b). \tag{4.1}$$

For large b, $I_0(bW) \sim \frac{\mathrm{e}^{bW}}{\sqrt{2\pi bW}}$, so that for the integral to converge, one needs

$$|\tilde{F}(s,b)| < \mathrm{e}^{-bW_0} \quad \text{with} \quad W_0 \simeq 2\mu, \tag{4.2}$$

which implies that

$$|1 - \mathrm{e}^{-n(b,s)/2}| < \mathrm{e}^{-bW_0/2}. \tag{4.3}$$

Thus, analyticity dictates that $n(b,s)$ must be bounded at least by an exponential.

Stronger (but model dependent) constraints arise provided one imposes that the elastic differential cross-section exhibits a "diffraction peak". That is, as in Sect. 2, let

$$F(s,t) \simeq f(s)\mathrm{e}^{\hat{b}(s)t}, \tag{4.4}$$

where $\hat{b}(s)$ is the so-called width of the diffraction peak, which has an observed (approximately logarithmic in) s dependence. Then,

$$\tilde{F}(s,b) \simeq \frac{\mathrm{i}f(s)}{4\pi s} \int \mathrm{d}q^2 J_0(bq)\mathrm{e}^{-\hat{b}(s)q^2} = \frac{-\mathrm{i}f(s)}{2\pi s}\left[\frac{1}{2\hat{b}(s)}\right]\mathrm{e}^{-\frac{b^2}{4\hat{b}(s)}}, \tag{4.5}$$

which requires a Gaussian fall off of the amplitude in the impact parameter b, with its scale determined by the width of the diffraction peak. In the Regge pole description,

$$\hat{b}(s) \sim \alpha' \ln(s/s_0) \qquad f(s) \sim \mathrm{i}\beta(s/s_0)^{1+\epsilon}, \tag{4.6}$$

and

$$\tilde{F}(s,b) \simeq \frac{\beta(s/s_0)^{\epsilon}}{4\pi(\alpha' s_0)\ln(s/s_0)}\mathrm{e}^{-\frac{b^2}{4\alpha'\ln(s/s_0)}}. \tag{4.7}$$

Classic mini-jet models [13,14] were constructed to describe the "hard" part of the cross-section as arising from the scattering of partons. In certain mini-jet models, where the impact parameter distribution is given by the Fourier transform of the proton form factor, a model we refer to as the form factor (FF) model [14], the following choice is made:

$$n(b,s) = \frac{\nu^2}{96\pi}(\nu b)^3 K_3(\nu b)[\sigma_{\text{soft}} + \sigma_{\text{jet}}]. \tag{4.8}$$

The modified Bessel functions of the third kind $K_\mu(z)$ are bounded by an exponential at large values of the argument, i.e.

$$K_\mu(z) \sim \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}\left\{1 + \mathcal{O}\left(\frac{1}{z}\right)\right\}.$$

We see then that while the FF formulation does (barely) satisfy the requirements of analyticity in the Lehmann ellipse, it lacks the observed shrinking of the diffraction peak, corresponding to an energy dependent near Gaussian fall off in b. Even if one were to introduce an ad hoc energy dependence (as is often the practice) instead of the constant scale parameter ν (as in the FF model), still one would be nowhere near the stronger Gaussian decrease at large impact parameters (i.e., for large b). This is one reason why the FF model (and its simple variants), albeit with jet cross-sections driving the rise of the cross-section, fails to provide an adequate description of the over all energy dependence of the total cross-section, unless further ad hoc modifications are introduced in its b dependence.

After this brief analysis of the virtues and deficiencies of several models, we turn our attention to the central result of this paper: an asymptotic sum rule which is essentially model independent.

5 An asymptotic sum rule

In the eikonal picture, the dimensionless "b-wave cross-sections" are given by [see (2.7)–(2.9)]

$$\frac{\mathrm{d}^2\sigma_{\text{el}}}{\mathrm{d}^2\mathbf{b}} = 1 - 2\eta(s,b)\cos\{2\delta_{\text{R}}(s,b)\} + \eta^2(s,b), \tag{5.1a}$$

$$\frac{\mathrm{d}^2\sigma_{\text{inel}}}{\mathrm{d}^2\mathbf{b}} = 1 - \eta^2(s,b), \tag{5.1b}$$

and

$$\frac{\mathrm{d}^2\sigma_{\text{tot}}}{\mathrm{d}^2\mathbf{b}} = 2[(1 - \eta(s,b)\cos\{2\delta_{\text{R}}(s,b)\}]. \tag{5.1c}$$

Equations (5.1c) show explicitly the maximum permissible rise for the different cross-sections conceded by unitarity. For complete absorption of "low" partial waves at asymptotic energies (which translates into $\eta(s,b) \to 0$ for $b \to 0$ and $s \to \infty$), one obtains the geometric limit (including the contribution from shadow scattering):

$$\frac{\mathrm{d}^2\sigma_{\text{el}}}{\mathrm{d}^2\mathbf{b}} = \frac{\mathrm{d}^2\sigma_{\text{inel}}}{\mathrm{d}^2\mathbf{b}} = \frac{1}{2}\frac{\mathrm{d}^2\sigma_{\text{tot}}}{\mathrm{d}^2\mathbf{b}} \to 1 \text{ for } b \to 0 \text{ and } s \to \infty. \tag{5.2}$$

Evidence for such a maximum rise (i.e., the validity of (5.2)) has been provided through various models, such as the resummed soft gluon models [6,7,11,12] and other models [8], all of which incorporate the observed rise in pp and $p\bar{p}$ total cross-sections. Our objective in the present section is to provide global predictions through sum rules over experimentally measurable quantities (such as $\frac{\mathrm{d}\sigma}{\mathrm{d}t}$). As we shall see later, the results of the sum rule are intimately related to this maximum rise.

We shall first derive an inequality for the dimensionless integral $I_0(s)$, defined as

$$I_0(s) = \frac{1}{2}\int_{-\infty}^{0} \mathrm{d}t\sqrt{\frac{\mathrm{d}\sigma}{\pi\mathrm{d}t}}, \tag{5.3}$$

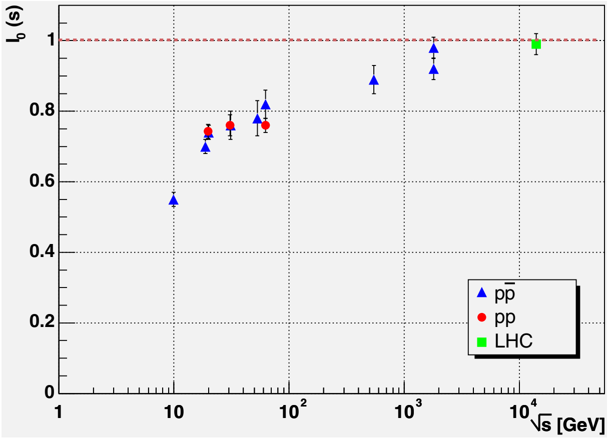

Fig. 1. A plot of $I_0(s)$ versus $\sqrt{s}$ using experimental data [15–23]. The last point is our extrapolation for LHC

Using (2.1) and (2.5), it is easy to show that

$$I_0(s) \geq [1 - \eta(s, o)]. \tag{5.4}$$

The above becomes an equality and in fact $I_0(s)$ equals 1 if we (initially) assume that at high energies
(i) the imaginary part of the scattering amplitude dominates the real part and that
(ii) the imaginary part does not change sign. Our sum rule thus reads

$$I_0(s) = \frac{1}{2}\int_{-\infty}^{0} \mathrm{d}t \sqrt{\frac{\mathrm{d}\sigma}{\pi \mathrm{d}t}} \to 1, \text{ for } s \to \infty. \tag{5.5}$$

$I_0(s)$ should rise from its threshold value $2|a_0|k \to 0$, where a_0 is the S-wave scattering length (complex for $p\bar{p}$) and k is the CM 3-momentum, to its asymptotic value 1 as s goes to infinity. In Fig. 1 , we show a plot of this integral for available data [15–23] on pp and $p\bar{p}$ elastic scattering for high energies[2].

Highest energy data at $\sqrt{s} = 1.8\,\mathrm{TeV}$ for $p\bar{p}$ from the Fermilab Tevatron give an encouraging value of 0.98 ± 0.03, demonstrating that indeed the integral is close to its asymptotic value of 1. We expect it to be even closer to 1 at the LHC (our extrapolation gives the value 0.99 ± 0.03 for LHC).

Having found the trend to its asymptotic value at the highest available energies, we may return to the question whether the two assumptions made to obtain the sum rule are really necessary. Presently, we know theoretically that in the forward direction ($t = 0$), the imaginary part must dominate the real part if the cross-section saturates the Froissart–Martin bound. That is, if at high energies [25] $\sigma_{\mathrm{tot}}(s) \to \text{Constant} \times \ln^2(s/s_0)$, then

$$\rho(s, 0) = \frac{\mathrm{Re}F(s, 0)}{\mathrm{Im}F(s, 0)} \to \frac{\pi}{\ln(s/s_0)} \to 0. \tag{5.6}$$

[2] Interesting new data on pp elastic scattering at $\sqrt{s} = 200\,\mathrm{GeV}$ have recently been published [24]. However, these data cover only the very forward region ($|t| < 0.019\,\mathrm{GeV}^2$) and hence are moot regarding our sum rule.

We can make a crude estimate of $\rho(s, 0)$ through (5.6). For the highest Tevatron energy, we find it to be about 0.2. Since the contribution of the real part to the integral only enters as $(1/2)\rho^2$, the neglect of the real part in the forward direction would affect the integrand by about 2%, i.e., well within the experimental errors which are about 3%.

If the total cross-section were to increase only as $\ln(s/s_0)$, the ratio of real to the imaginary part would still go down to zero as in (5.6), albeit with a smaller constant $\pi/2$. Thus, for rising cross-sections, we are assured of the correctness of our first assumption in the forward direction. For non-forward directions, we have no direct evidence experimentally. However, since the overall differential cross-section would be decreasing (for $t \neq 0$) as a function of s, their contribution to $I_0(s)$ is less important. It is for exactly the same reason that the second assumption i.e., the absence of zeros in the imaginary part of the non-forward amplitude is not really necessary. So long as any possible such zeroes remain at some finite values of (negative) t, they would not significantly upset the sum rule. The satisfaction of our sum rule a posteriori justifies this claim.

6 Discussion of results and conclusions

There are some interesting and significant consequences which follow from the above analysis. First, since the asymptotic value is reached from below, we may bound η, the inelasticity in the central ($b = 0$) region. For example, even at $\sqrt{s} = 100\,\mathrm{GeV}$, absorption is not complete but only about 80%, giving us a quantitative understanding of where the onset of "high energy" lies.

Another deduction concerning universality of the above result may be made. That is, the central value of inelasticity should approach zero for the scattering of all hadrons (at least for all hadrons made of light quarks). Such a result follows naturally from QCD if we recall that both for nucleons as well as for mesons, half the hadronic energy is carried by glue [26,27]. In QCD, such an equipartition of energy has been derived rigorously to hold for all hadrons which are bound states of massless quarks [28]. Since all available high energy elastic scattering data are for nucleons and light mesons, all of which are made of the very light quarks, we have excellent support from QCD for equipartition. If one couples this with the notion that the rise of the cross-section is through gluon–gluon scattering, which is flavor independent, the asymptotic equality of (the rise in) all hadronic cross-sections and an eventual flavor independence automatically emerges.

Of course, the approach to asymptotes would not be the same for nucleon–nucleon and meson–nucleon scatterings. It is unfortunate that data for πN scattering are available only up to $\sqrt{s} = 20\,\mathrm{GeV}$, which is far from asymptotic. In fact, for this channel, $I_0(s)$ is only about 0.6 at the highest energy measured so far. Since the same asymptotic value of 1 for this integral should be reached for all hadrons, the rise with energy must be even more dramatic for meson–nucleon scattering. In principle, such a test for RHIC and LHC may be feasible through Bjorken's suggestion [29] of converting an incident proton into a pion by isolating the one pion exchange contribution via tagging or triggering on a leading neutron or Δ^{++}.

It appears reasonable to extend our analysis to NA or even AB elastic scatterings, where A, B are nuclei. Also for these processes, at very high energies, we expect from QCD that the central inelasticity should approach zero and hence (modulo possible complications were $\rho(s,t)$ to be anomalously large), $I_0(s)$ should again asymptotically go to 1. For illustrative purposes, let us consider the following very simple expression which incorporates the sum rule:

$$|F_{AB}(s,t)| = I_0(s)\mathcal{B}_{AB}(s)\mathrm{e}^{(1/2)\mathcal{B}_{AB}(s)t}, \tag{6.1}$$

Parametrizations of the above form (which underestimate the large t contributions by ignoring the secondary slopes) are routinely used. However, what is new here is that since $I_0(s)$ goes to 1 in the asymptotic limit, the prefactor would, in the same limit, become equal to the diffraction width $\mathcal{B}_{AB}(s)$. Physically, it says that unitarity correlates and limits how large the amplitude can be, as a function of the energy, to how fast it decreases, as a function of the momentum transfer. If (6.1) holds, we may use the optical theorem to obtain the approximate expression

$$I_0(s) = \left[\frac{\sigma_{\mathrm{tot}}(s)}{4\pi\mathcal{B}_{AB}(s)}\right]\left[1 + (1/2)\rho^2_{AB}(s,0)\right]. \tag{6.2}$$

Under the same assumption, we would have

$$\frac{\sigma_{\mathrm{el}}(s)}{\sigma_{\mathrm{tot}}(s)} \approx \left[\frac{I_0(s)}{4}\right]\left[1 + \rho^2_{AB}(s,0)\right]. \tag{6.3}$$

For the highest Tevatron energy $\sqrt{s} = 1.8\,\mathrm{TeV}$, (6.3) would estimate the elastic to total ratio to be about 0.25 in excellent accord with the experimental value (0.25 ± 0.02) [30].

Future experiments from RHIC and LHC should be able to test our sum rule predictions for *pp* and other elastic channels. For this purpose, it would be useful that in the future, experimentalists present values of $I_0(s)$ directly from their experimental data, obviating thereby interpolations (such as those carried out by us to obtain Fig. 1).

References

1. M. Froissart, Phys. Rev. **123**, 1053 (1961)
2. A. Martin, Phys. Rev. **129**, 1432 (1963)
3. A. Martin, F. Cheung, Analyticity properties and bounds on scattering amplitudes (Gordon and Breach, Science Publishers, New York 1970)
4. S.W. MacDowell, A. Martin, Phys. Rev. B **137**, 960 (1965)
5. Particle Data Group, S. Eidelman et al., Phys. Lett. B **592**, 1 (2004)
6. A. Grau, G. Pancheri, Y.N. Srivastava, Phys. Rev. D **60**, 114020 (1999); A. Corsetti, A. Grau, G. Pancheri, Y.N. Srivastava, Phys. Lett. B **382**, 282 (1996)
7. A. Corsetti, A. Grau, R.M. Godbole, G. Pancheri, Y.N. Srivastava, hep-ph/9605254; R.M. Godbole, A. Grau, G. Pancheri, Y.N. Srivastava, Invited talk at the International Workshop on QCD, Martina Franca, Italy, June, 2001; hep-ph/0205196; A. Grau, R.M. Godbole, G. Pancheri, Y.N. Srivastava, hep-ph/0408355 (submitted to Phys. Rev. for publication)
8. M.M. Block, E.M. Gregores, F. Halzen, G. Pancheri, Phys. Rev. D **60**, 054024 (1999); M.M. Block, Nuc. Phys. B (Proc. Suppl.) **126**, 76 (2004); M.M. Block, K. Kang, hep-ph/0302146
9. A. Donnachie, P.V. Landshoff, Phys. Lett. B **550**, 160 (2002)
10. T. Gaisser, F. Halzen, Phys. Rev. Lett. **54**, 1754 (1985)
11. A. Grau, S. Pacetti, G. Pancheri, Y.N. Srivastava, Nuc. Phys. B (Proc. Suppl.) **126**, 84 (2004)
12. R.M. Godbole, A. Grau, G. Pancheri, Y.N. Srivastava, Nuc. Phys. B (Proc. Suppl.) **126**, 94 (2004)
13. D. Cline, F. Halzen, J. Luthe, Phys. Rev. Lett. **31**, 491 (1973); G. Pancheri, C. Rubbia, Nuc. Phys A **418**, (1984); G. Pancheri, Y. Srivastava, Phys. Lett. B **158**, 402 (1986); M.M. Block, F. Halzen, B. Margolis, Phys. Rev. D **45**, 839 (1992); M. Treleani, L. Amettler, Int. J. Mod. Phys. A **3**, 521 (1988)
14. L. Durand, H. Pi, Phys. Rev. Lett. **58**, 303 (1987); A. Capella, J. Kwiecinsky, J. Tran Thanh, Phys. Rev. Lett. **58**, 2016 (1987)
15. F. Abe et al., Phys. Rev. D **50**, 5518 (1994)
16. N.A. Amos et al., Phys. Lett. B **247**, 127 (1990)
17. M. Bozzo et al., Phys. Lett. B **155**, 197 (1985); B **147**, 392 (1984)
18. N.A. Amos et al., Nucl. Phys. B **262**, 689 (1985)
19. A. Breakstone et al., Nucl. Phys. B **248**, 253 (1984); Phys. Rev. Lett. **54**, 2160 (1985)
20. U. Amaldi, K.R. Schubert, Nucl. Phys. B **166**, 013 (1980)
21. C. Akerlof et al., Phys. Rev. D **14**, 2864 (1976)
22. D. Ayres et al., Phys. Rev. D **15**, 3105 (1977)
23. Z. Asad et al., Phys. Lett. B **108**, 51 (1982)
24. S. Bültmann et al., nucl-ex/0305012
25. N.N. Khuri, T. Kinoshita, Phys. Rev. B **137**, 720 (1965)
26. W.K. Ellis, W.J. Stirling, W.R. Webber, QCD and collider physics (Cambridge University Press, New York 1996)
27. V. Barger, R.J. Phillips, Collider physics (Addison Wesley Publishing Co., California 1987)
28. Y.N. Srivastava, A. Widom, Phys. Rev. D **63**, 077502 (2001)
29. J.D. Bjorken, Nuc. Phys. B (Proc. Suppl.) **71**, 484 (1999)
30. N.A. Amos et al., Phys. Rev. Lett. **63**, 2784 (1989)